Motion In A Plane

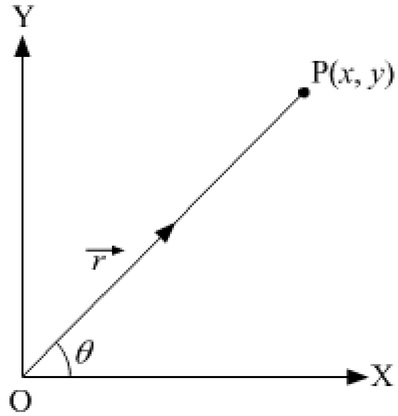
Scalars and Vectors; Multiplication of Vectors by Real Numbers

Scalars vs. Vectors

Scalars	Vectors
A scalar quantity has magnitude only.	A vector quantity has both magnitude and direction.
Scalar quantities can be added, subtracted, multiplied and divided just like ordinary numbers, i.e., scalars are subjected to simple arithmetic operations.	Vectors cannot be added, subtracted or multiplied following simple arithmetic rules. Arithmetic division of vectors is not possible at all.
Example: Mass, volume, time, distance, speed, work, temperature, etc.	Example: Displacement, velocity, acceleration, force, etc.

Position Vector

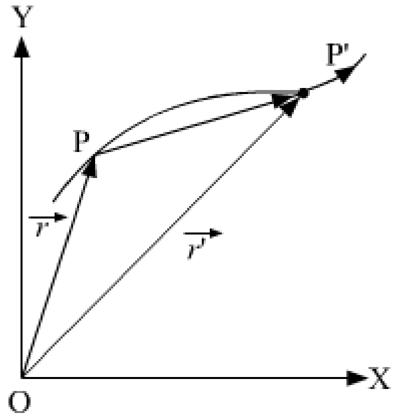
The position vector of a point in a coordinate system is the straight line that joins the origin and the point.



The magnitude of a vector is the length of the straight line. Its direction is along the angle θ from the positive x-axis.

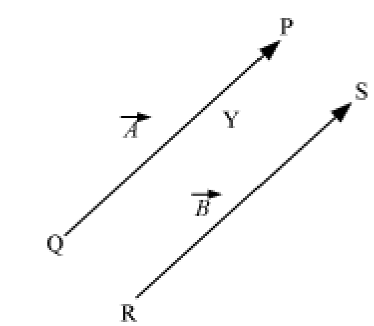
Displacement Vector

Displacement vector is the straight line joining the initial and the final position.

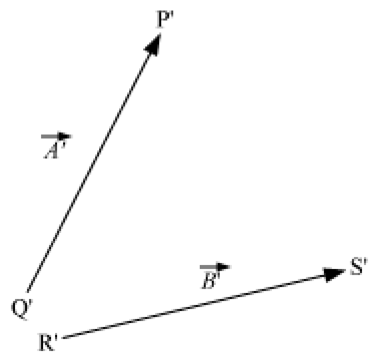


Equality of Vectors

Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal only if they have the same magnitude and the same direction.



$\vec{A}$ and $\vec{B}$ are equal



$\vec{A'}$ and $\vec{B'}$ are unequal vectors; their magnitudes are same but directions are not same.

Negative vector

Negative vector is a vector whose magnitude is equal to that of a given vector, but whose direction is opposite to that of the given vector.

Zero vector

Zero vector is a vector whose magnitude is zero and have an arbitrary direction.

Resultant vector

The resultant vector of two or more vectors is a vector which produces the same effect as produced by the individual vectors together.

Multiplication of Vectors by Real Numbers

- Multiplication of a vector $\vec{A}$ with a positive number k only changes the magnitude of the vector keeping its direction unchanged.

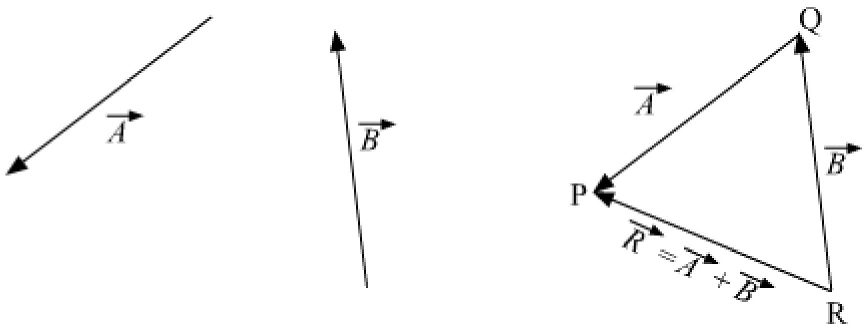
  $\left|k\vec{A}\right| = k\left|\vec{A}\right|$ if $k > 0$

- Multiplication of a vector $\vec{A}$ with a negative number $-k$ gives a vector $-k\vec{A}$ in the opposite direction.

Addition and Subtraction of Vectors

Addition of Vectors: Triangle Method

- The given vectors $\vec{A}$ and $\vec{B}$ have to be arranged head to tail, keeping their directions unchanged.

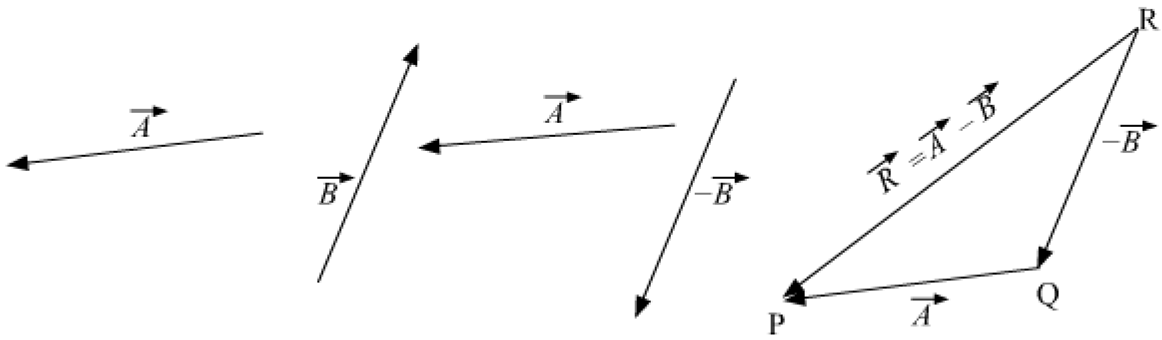


- The line PR , joining the ending point of $\vec{A}$ and the starting point of $\vec{B}$, represents a vector $\vec{R}$ (resultant vector) that is the sum of the vectors $\vec{A}$ and $\vec{B}$.
  i.e., $\vec{R} = \vec{A} + \vec{B}$

- Vector addition obeys commutative law and associative law.
  i.e., $\vec{A} + \vec{B} = \vec{B} + \vec{A}$ and $\left(\vec{A} + \vec{B}\right) + \vec{C} = \vec{A} + \left(\vec{B} + \vec{C}\right)$

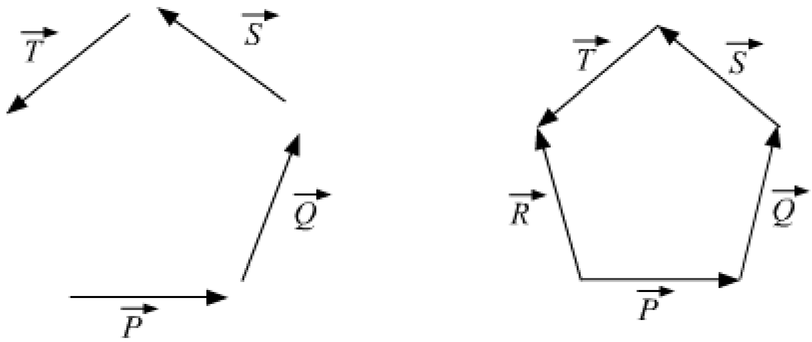
Subtraction of Vectors

- The difference between two vectors $\vec{A}$ and $\vec{B}$ is defined as the sum of two vectors $\vec{A}$ and $-\vec{B}$.
  i.e., $\vec{R} = \vec{A} - \vec{B} = \vec{A} + \left(-\vec{B}\right)$

- 

Polygon law of vector

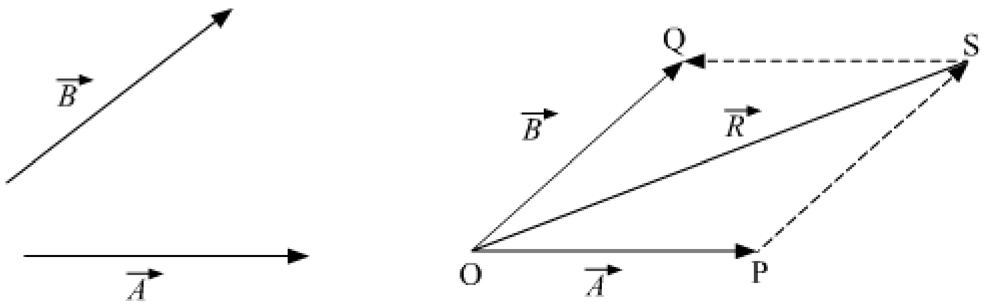
- According to this law, if a number of vectors acting in a plane are represented in magnitudes and directions by the sides of an open polygon taken in order, then the resultant vector is represented in magnitude and direction by the closing side of the polygon taken in the opposite order. The direction of the resultant vector is from the starting point of the first vector to the end point of the last vector.



- Vectors $\vec{P}$, $\vec{Q}$, $\vec{S}$ and $\vec{T}$ are placed in order to represent an incomplete polygon. Their resultant vector $\vec{R}$ represents the closing side of the polygon.

Parallelogram Method of Vector Addition

- The given vectors $\vec{A}$ and $\vec{B}$ have to be arranged, keeping their directions unchanged such that their starting point is a common point O.



- If a parallelogram OQSP is drawn with these two vectors as its sides, then the diagonal OS is the sum $\vec{R}$ (resultant vector) of the two vectors.
- The length of the diagonal is the magnitude of the resultant vector and its direction is along the diagonal OS.
- The magnitude of the resultant vector *R* is given by

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

Special cases

(1) If $\vec{A}$ and $\vec{B}$ are perpendicular to each other, then $\theta = 90^\circ$

$$R = \sqrt{A^2 + B^2 + 2ABcos90^\circ}$$
$$\Rightarrow R = \sqrt{A^2 + B^2} \quad (\because cos90^\circ = 0)$$

$$\text{Now, } \tan\theta = \frac{B \sin 90^\circ}{A + B\cos 90^\circ}$$
$$\Rightarrow \tan\theta = \frac{B}{A}$$
$$\Rightarrow \theta = \tan^{-1}\left(\frac{B}{A}\right)$$

(2) If $\vec{A}$ and $\vec{B}$ are parallel to each other, then $\theta = 0^\circ$

$$R = \sqrt{A^2 + B^2 + 2AB\cos 0^\circ}$$
$$\Rightarrow R = \sqrt{A^2 + B^2 + 2AB}$$
$$\Rightarrow R = A + B$$

$$\text{Now,}\ \tan\theta = \frac{B\sin 0^\circ}{A + B\cos 0^\circ}$$

$$\Rightarrow \theta = 0 \qquad (\because\ \sin 0^\circ = 0)$$

(3) If $\vec{A}$ and $\vec{B}$ are antiparallel to each other, then θ =180°

$$R = \sqrt{A^2 + B^2 + 2AB\cos 180^\circ}$$

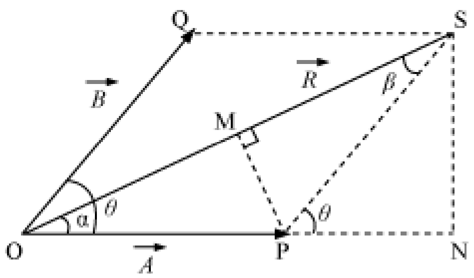
$$\Rightarrow R = \sqrt{A^2 + B^2 - 2AB} \qquad (\because\ \cos 180^\circ = -1)$$

$$\Rightarrow R = A - B$$

$$\text{Now,}\ \tan\theta = \frac{B\sin 180^\circ}{A + B\cos 180^\circ}$$

$$\Rightarrow \theta = 0^\circ \qquad (\because \sin 180^\circ = 0)$$

Addition of Vectors by Analytical Method



Let $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ represent the two vectors $\vec{A}$ and $\vec{B}$, making an angle θ.

$$\therefore \vec{R} = \vec{A} + \vec{B}$$

For right-angled triangle ONS,

$OS^2 = ON^2 + SN^2$

However, $ON = OP + PN = A + B\cos\theta$

$SN = B\sin\theta$

$OS^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$

$\Rightarrow R^2 = A^2 + B^2 + 2AB\cos\theta$ … (i)

In ΔOSN, SN = OS sin α= R sin α

In ΔPSN, SN = PS sinθ= B sinθ

$$\Rightarrow \frac{R}{\sin\theta} = \frac{B}{\sin\alpha} \qquad ...(ii)$$

Similarly,

PM = A sinα= B sinβ

$$\frac{A}{\sin\beta} = \frac{B}{\sin\alpha} \qquad ...(iii)$$

Combining equations (ii) and (iii), we obtain

$$\Rightarrow \frac{R}{\sin\theta} = \frac{A}{\sin\beta} = \frac{B}{\sin\alpha} \qquad ...(iv)$$

Using equation (iv), we obtain

$$\sin\alpha = \frac{B}{R}\sin\theta \qquad ...(v)$$

Where 'R' is given by equation (i)

$$\Rightarrow \tan\alpha = \frac{\text{SN}}{\text{OP} + \text{PN}} = \frac{B\sin\theta}{A + B\cos\theta} \qquad ...(vi)$$

Equation (i) gives the magnitude of the resultant and equation (v) and (vi) its directions.

Equation (i) is known as the law of cosines and equation (iv) as the law of sines.

Problems Based on Addition of Vectors by Analytical Method

Example − Two forces 10 N and 15 N are acting at an angle of 120°between them. Find the resultant force in magnitude and direction.

Solution

Here, A = 10 N, B = 15 N

θ = 120°; R =?; α= ?

$$R=\sqrt{A^2+B^2+2AB\cos\theta}$$

$$\Rightarrow R=\sqrt{(10)^2+(15)^2+2\times10\times15\cos120^\circ}$$

$$\Rightarrow R=\sqrt{100+225+300\left(-\frac{1}{2}\right)}$$

$$R=\sqrt{100+225-150}$$

$$\Rightarrow R=\sqrt{175}$$

$\Rightarrow R$= 13.2 N

$$\tan\alpha=\frac{B\sin\theta}{A+B\cos\theta}$$

$$\Rightarrow \tan\alpha=\frac{15\sin120^\circ}{10+15\cos120^\circ}$$

$$\Rightarrow \tan\ \alpha=\frac{15\times\frac{\sqrt{3}}{2}}{10+15\times\left(-\frac{1}{2}\right)}=\frac{15\times0.87}{10-7.5}$$

$$=\frac{13.05}{2.5}=5.22$$

$$\Rightarrow \tan\ \alpha=5.22$$

$$\Rightarrow \alpha=\tan^{-1}(5.22)$$

$$=\alpha=79.15^\circ$$

Resolution of Vectors

A unit vector is a vector of unit magnitude and points towards a particular direction.

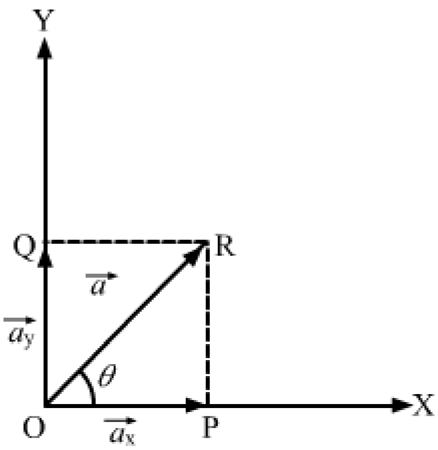
- Unit vector can be expressed as $\widehat{a} = \frac{\vec{a}}{|\vec{a}|}$

- $\hat{i}, \hat{j}$ and $\hat{k}$ are three special unit vectors along X, Y, and Z axes respectively.

Resolution of Vector in Rectangular Components (in Two Dimensions)

- The process of splitting a vector into rectangular components is called resolution of vector.

- The components of a vector are found by projecting the vector on the axes of a rectangular coordinate system. The coordinate system can be considered according to our convenience.



- $\vec{a}_x$ and $\vec{a}_y$ are the components of vector $\vec{a}$ along X-axis and Y-axis respectively.

From triangle law of vector addition, we have

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$$

$$\text{or } \vec{a} = \vec{a}_x + \vec{a}_y \qquad \left[\because \overrightarrow{OP} = \vec{a}_x \text{ and } \overrightarrow{OQ} = \vec{a}_y\right]$$

Let $\hat{i}$ and $\hat{j}$ be the unit vectors along X-axis and Y-axis respectively.

$$\therefore \vec{a}_x = a_x \hat{i} \text{ and } \vec{a}_y = a_y \hat{j}$$

Hence, $\vec{a} = a_x \hat{i} + a_y \hat{j}$

From right-angled triangle ORP, $a_x = a \cos\theta$ and $a_y = a \sin\theta$

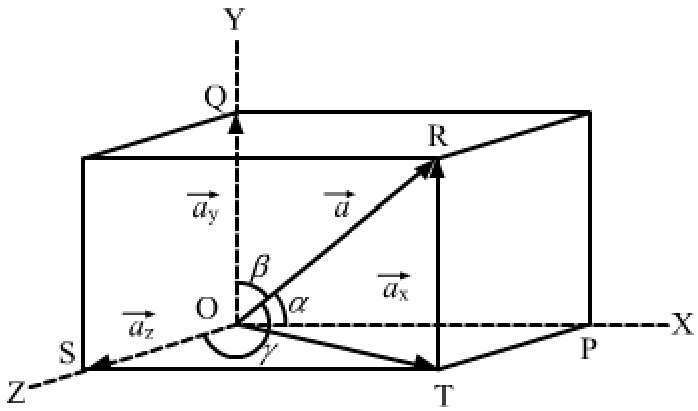
Thus, the magnitudes of the components are

$$\boldsymbol{a_x = a \cos\theta \text{ and } a_y = a \sin\theta}$$

Therefore, if the components of a vector are known, then its magnitude and direction can be determined by using the following equations.

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{PR}{OP} = \frac{a_y}{a_x}$$

Rectangular Components of a Vector in Three Dimensions



Using triangle law of vectors,

$\overrightarrow{OR} = \overrightarrow{OT} + \overrightarrow{TR}$

Using parallelogram law of vectors,

$\overrightarrow{OT} = \overrightarrow{OS} + \overrightarrow{OP}$

$\therefore \overrightarrow{OR} = \overrightarrow{OS} + \overrightarrow{OP} + \overrightarrow{TR}$

$= \overrightarrow{OS} + \overrightarrow{OP} + \overrightarrow{OQ} \qquad \left[\because \overrightarrow{TR} = \overrightarrow{OQ}\right]$

or $\vec{a} = \vec{a}_z + \vec{a}_x + \vec{a}_y$

Let $\vec{a}_z = a_z\hat{k},\ \vec{a}_x = a_x\hat{i},\ \vec{a}_y = a_y\hat{j}$

$\therefore \vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

If, α, β, and γ are the angles which $\vec{a}$ makes with X, Y and Z axes respectively, then

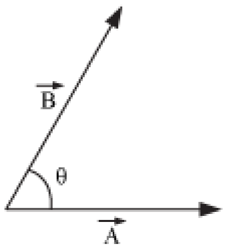
$a_x = a\cos\alpha$, $a_y = a\cos\beta$, $a_z = a\cos\gamma$

And, $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$

Scalar Product

- Scalar product of two vectors $\vec{A}$ and $\vec{B}$ is given by

$\vec{A}\cdot\vec{B} = AB\cos\theta$



- It is also known as dot product.

- The result of the scalar product of two vectors is a scalar quantity.
- When two vectors are parallel, $\theta = 0°$, $\cos 0° = 1$

$$\therefore\ \vec{A}\,.\,\vec{B} = AB\cos\ 0^{\circ} = AB$$

- ∴ A⟶.B⟶=ABcos 0°=AB
- For unit vectors, $\hat{i}\cdot\hat{i} = |\hat{i}||\hat{i}|\cos 0°$

  $= 1\times 1\times 1 = 1$

  Similarly, $\hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$
- When two vectors are perpendicular, $\theta = 90°$, $\cos 90° = 0$

  $\therefore \vec{A}\cdot\vec{B} = AB\cos 90° = 0$

  It means the dot product of two perpendicular vectors is zero.
- For unit vectors, $\hat{i}\cdot\hat{j} = |\hat{i}||\hat{j}|\cos 90°$

  OR

  $\hat{i}\cdot\hat{j} = 1\times 1\times 0 = 0$

  Similarly, $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

  Properties of Scalar Product of two vectors

- Scalar product of two vectors is commutative, i.e.,

  $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
- Scalar product is distributive, i.e.,

  $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

- Scalar product of a vector with itself gives the square of its magnitude, i.e.,

$$\vec{A} \cdot \vec{A} = A^2$$

Dot Product in Cartesian Coordinates

Let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\text{and } \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

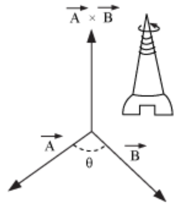
$$\vec{A} \cdot \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \cdot (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$\begin{aligned}\vec{A} \cdot \vec{B} &= A_x\hat{i} \cdot (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &\quad + A_y\hat{j} \cdot (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &\quad + A_z\hat{k} \cdot (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= A_xB_x(\hat{i} \cdot \hat{i}) + A_xB_y(\hat{i} \cdot \hat{j}) + A_xB_z(\hat{i} \cdot \hat{k}) + \\ &\quad A_yB_x(\hat{j} \cdot \hat{i}) + A_yB_y(\hat{j} \cdot \hat{j}) + A_yB_z(\hat{j} \cdot \hat{k}) + A_zB_x \\ &\quad (\hat{k} \cdot \hat{i}) + A_zB_y(\hat{k} \cdot \hat{j}) + A_zB_z(\hat{k} \cdot \hat{k}) \\ &= A_xB_x(1) + A_xB_y(0) + A_xB_z(0) + A_yB_x(0) \\ &\quad + A_yB_y(1) + A_yB_z(0) + A_zB_x(0) + A_zB_y(0) + A_zB_z(1)\end{aligned}$$

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Vector Product

- The magnitude of the vector product of two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of the magnitude of the vectors $\vec{A}$ and $\vec{B}$ and sine of the smaller angle between them.



$\vec{A} \times \vec{B} = AB\sin\theta\hat{n}$ ... (i)

Here, $\hat{n}$ is the unit vector perpendicular to both $\vec{A}$ and $\vec{B}$.

- The cross product of two vectors $\vec{A}$ and $\vec{B}$ is a vector, which is at right angles to both $\vec{A}$ and $\vec{B}$ and points in the direction in which a right-handed screw will advance.

- The vector product of two like vectors is zero.

  Example:

  $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = (1)(1)\sin 0°(\hat{n}) = 0$

- $\hat{i} \times \hat{j} = (1)(1)\sin 90°(\hat{k}) = \hat{k}$

  Here, $\hat{k}$ is the unit vector perpendicular to the plane of $\hat{i}$ and $\hat{j}$ and is in the direction in which a right-handed screw will move, when rotated from $\hat{i}$ to $\hat{j}$.

  Also, $\hat{j} \times \hat{i} = (1)(1)\sin 90°(-\hat{k}) = -\hat{k}$

  Similarly,

  $\hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i}$ and $\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$

- Let $\vec{A}$ and $\vec{B}$ be two vectors.

  $\vec{A} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

  $\vec{B} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

  Then,

$$\vec{A} \times \vec{B} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) \times (x_2\hat{i} + y_2\hat{j} + z_2\hat{k})$$

$$\vec{A} \times \vec{B} = (y_1z_2 - z_1y_2)\hat{i} + (z_1x_2 - x_1z_2)\hat{j} + (x_1y_2 - y_1x_2)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

Properties of Vector Product

- The cross product of a vector with itself is a null vector.

$$\vec{A} \times \vec{A} = (A)(A)\sin 0^\circ \hat{n} = \vec{0}$$

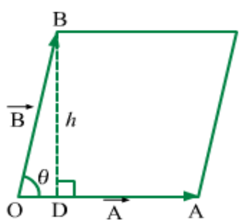
- The cross product of two vectors does not obey commutative law.

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

- The cross product of vectors obeys the distributive law.

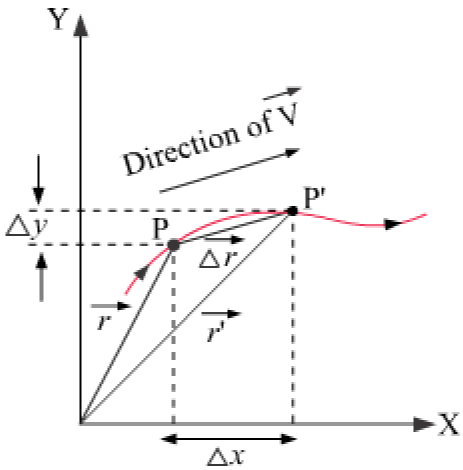
$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

- If the vectors $\vec{A}$ and $\vec{B}$ represent the two adjacent sides of a parallelogram, the magnitude of cross product of $\vec{A}$ and $\vec{B}$ will represent the area of the parallelogram.



- Area of parallelogram ABCD = $\left|\vec{A} \times \vec{B}\right|$

Motion in a Plane



Displacement $\left(\Delta \vec{r}\right)$

Suppose the particle is at point **P** at time t and **P'** at time t'. The displacement is

$$\Delta \vec{r} = \vec{r'} - \vec{r}$$

In component form,

$$\Delta \vec{r} = \left(x'\hat{i} + y'\hat{j}\right) - \left(x\hat{i} + y\hat{j}\right)$$

$$\Rightarrow \Delta \vec{r} = \Delta x\hat{i} + \Delta y\hat{j}$$

Where $\Delta x = x' - x$

$\Delta y = y' - y$

Velocity $\left(\Delta \vec{v}\right)$

$$\vec{v} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\Delta x\hat{i} + \Delta y\hat{j}}{\Delta t}$$

$$= \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j}$$

$$= \Delta \vec{v} = \Delta \vec{v}_x \hat{i} + \Delta \vec{v}_y \hat{j}$$

The instantaneous velocity is given by the limiting value of the average velocity as the time interval approaches zero i.e.,

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

Acceleration $\left(\vec{a}\right)$

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\Delta\left(v_x \hat{i} + v_y \hat{j}\right)}{\Delta t}$$

$$= \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j}$$

$$= \vec{a} = a_x \hat{i} + a_y \hat{j}$$

The instantaneous acceleration is the limiting value of the average acceleration as the time interval approaches zero i.e.,

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t}$$

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta v_x \hat{i} + \Delta v_y \hat{j}}{\Delta t}$$

$$\vec{a} = \hat{i} \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} + \hat{j} \lim_{\Delta t \to 0} \frac{\Delta v_y}{\Delta t}$$

$$\vec{a} = a_x \hat{i} + a_y \hat{j}$$

$$\text{Where, } a_x = \frac{dv_x}{dt},\ a_y = \frac{dv_y}{dt}$$

Motion in a Plane and Relative Velocity in 2D

Suppose that an object is moving in *x*–*y* plane and its acceleration $\bar{a}$ is constant.

Let

$\vec{v}_0 \rightarrow$ Initial velocity of the object at time $t = 0$

$\vec{v} \rightarrow$ Final velocity of the object at time $t = t$

Then,

$$\vec{a} = \frac{\vec{v} - \vec{v}_0}{t - 0} = \frac{\vec{v} - \vec{v}_0}{t}$$

$$\Rightarrow \vec{v} = \vec{v}_0 + \vec{a}\, t \text{ ......(i)}$$

In terms of components,

$v_x = v_{0x} + a_x t$

$v_y = v_{0y} + a_y t$

Let $\vec{r_0}$ be the position vector of the particle at time 0 and $\vec{r}$ be the position vector of the particle at time t.

The displacement is the average velocity multiplied by the time interval.

$$\vec{r} - \vec{r}_0 = \left(\frac{\vec{v} + \vec{v}_0}{2}\right)t = \left(\frac{\vec{v}_0 + \vec{a}\,t + \vec{v}_0}{2}\right)t$$

$$\vec{r} - \vec{r}_0 = \vec{v}_0 t + \frac{1}{2}\vec{a}\,t^2$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}\,t^2$$

In component form:

$$\left.\begin{aligned} x &= x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \\ y &= y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \end{aligned}\right]$$

Motion in a plane can be treated as two separate simultaneous one-dimensional motions with constant acceleration along two perpendicular directions.

Relative Velocity in Two Dimensions

Suppose that two objects **A** and **B** are moving with velocities $\vec{v}_A$ and $\vec{v}_B$. Then, velocity of object **A** relative to that of **B** is

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

And the velocity of object **B** relative to that of **A** is

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$

and $\left|\vec{v}_{AB}\right| = \left|\vec{v}_{BA}\right|$

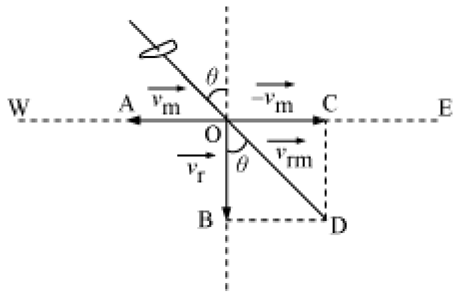
- **A special case − Relative velocity of rain with respect to the moving man**

Let

$\vec{v}_m$ →Velocity of man walking West represented by $\overline{OA}$

$\vec{v}_r$ → Velocity of the rain falling downwards

$\vec{v}_{rm}$ → Relative velocity of rain with respect to man



From the given figure, it is evident that $\overrightarrow{OD}$ represents the relative velocity of rain with respect to the man.

$$\therefore v_{rm} = \sqrt{v_r^2 + v_m^2 + 2v_r v_m \cos 90^\circ}$$

$$= \sqrt{v_r^2 + v_m^2}$$

$$\text{and } \tan\theta = \frac{BD}{OB} = \frac{v_m}{v_r}$$

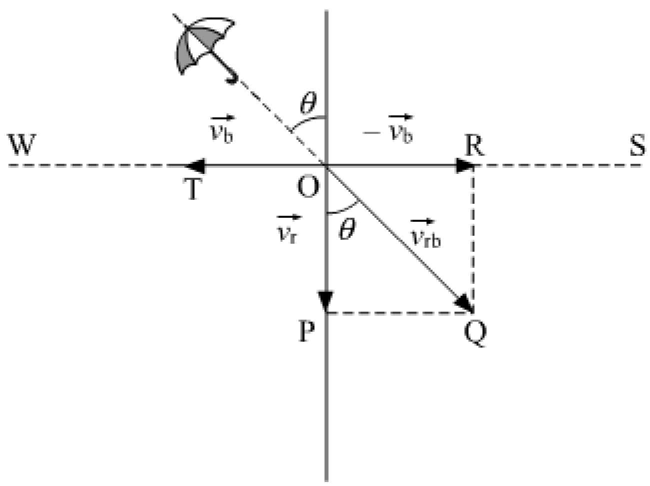
$$\Rightarrow \theta = \tan^{-1}\left(\frac{v_m}{v_r}\right)$$

Note

If the man wants to protect himself from rain, then he should hold an umbrella at an angle $\theta = \tan^{-1}\left(\frac{v_m}{v_r}\right)$ towards his motion with the vertical.

Example − A boy is riding a bicycle with a speed of 10 ms^{-1} from East to West direction. Rain falls vertically with a speed of 30 ms^{-1}. What is the direction in which he should hold his umbrella?

Solution



Velocity of rain, $\overrightarrow{v}_r = \left(\overrightarrow{OP}\right)$ = 30 ms^{-1} downwards

Velocity of the bicycle, $\overrightarrow{v}_b = \left(\overrightarrow{OT}\right)$ = 10 ms^{-1} towards West

The boy can protect himself from rain if he holds his umbrella in the direction of relative velocity of rain with respect to his bicycle.

The relative velocity of rain with respect to the bicycle $\overrightarrow{v}_{rb}$ will be the resultant of $\overrightarrow{v}_r$ and $-\overrightarrow{v}_b$.

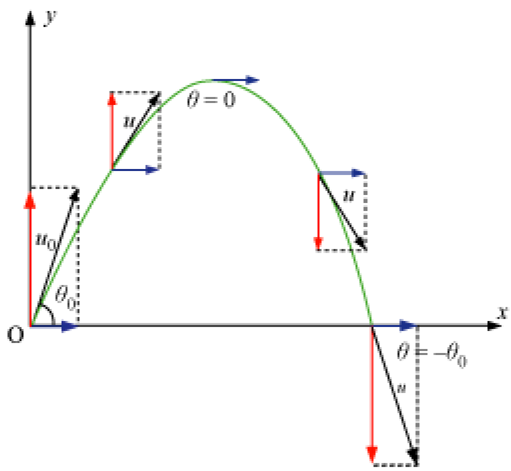
$$\tan\theta = \frac{PQ}{OP} = \frac{v_b}{v_r} = \frac{10}{30} = 0.333$$

$\theta = \tan^{-1}(0.333) = 18°43'$ with the vertical towards the West

Projectile Motion

- The motion of a projectile may be thought of as the result of horizontal and vertical components.
- Both the components act independently.

Projectile Given Angular Projection



- **Equation of the path of a projectile** − Suppose at any time t, the object is at point R (x, y).

For motion along the horizontal direction, the acceleration a_x is zero. The position of the object at any time t is given by,

$$x = x_0 + u_x t + \frac{1}{2} a_x t^2 \qquad \text{(i)}$$

Here, $x_0 = 0$, $u_x = u \cos \theta$ and $a_x = 0$.

($\because$ Velocity of an object in the horizontal direction is constant.)

Putting these values in equation (i), we get:

$$x = ut \cos\theta + \frac{1}{2}(0)t^2$$

$\Rightarrow x = ut \cos \theta$

$$\Rightarrow t = \frac{x}{u \cos\theta} \qquad \ldots\ldots \text{(ii)}$$

For motion along the vertical direction, the acceleration a_y is $-g$.

The position of the object at any time t along the vertical direction is given by,

$$y = y_0 + u_y t + \frac{1}{2} a_y t^2 \qquad \text{(iii)}$$

Here, $y_0 = 0,\ u_y = u \sin\theta,\ a_y = -g$

$$y = u \sin\theta t + \frac{1}{2}(-g)t^2$$

$$\therefore \boxed{y = u \sin\theta t - \frac{1}{2} g t^2}$$

Putting the value of t from equation (ii), we get:

$$y = u \sin\theta \left(\frac{x}{u\cos\theta}\right) - \frac{1}{2} g \left(\frac{x}{u\cos\theta}\right)^2$$

$\Rightarrow \quad y = x\tan\theta - \left(\frac{1}{2}\frac{g}{u^2\cos^2\theta}\right)x^2$

This is an equation of a parabola. Hence, the path of the projectile is a parabola.

It is the total time for which the object is in flight and is denoted by *T*.

Total time of flight = Time of ascent + Time of descent

$\therefore T = t + t = 2t$ [$\because$ Time of ascent = Time of descent = *t*]

$\Rightarrow \quad t = \frac{T}{2}$

At the highest point *H*, the vertical component of velocity becomes zero. For vertical motion of the object (from 0 to *H*),

$$u_y = u\sin\theta,\ a_y = -g, t = \frac{T}{2},\ v_y = 0$$
$$\because v_y = u_y + a_y t$$
$$\therefore 0 = u\sin\theta + (-g)\frac{T}{2}$$

$$\therefore \boxed{T = \frac{2u\sin\theta}{g}}$$

- **Maximum height** – It is the maximum height reached by the projectile and is denoted by *h*.

  For vertical upward motion from 0 to *H*,

  $$u_y = u\sin\theta,\ a_y = -g,\ y_0 = 0,\ y = h,\ t = \frac{T}{2} = \frac{u\sin\theta}{g}$$

  Using the relation $y = y_0 + u_y t + \frac{1}{2}a_y t^2$, we get:

  $$h = 0 + u\sin\theta \times \left(\frac{u\sin\theta}{g}\right) + \frac{1}{2}(-g)\left(\frac{u\sin\theta}{g}\right)^2$$

$\Rightarrow$ $h = \frac{u^2}{g}\sin^2\theta - \frac{1}{2}\frac{u^2\sin^2\theta}{g} = \frac{u^2\sin^2\theta}{2g}$

That is, $\boxed{h = \frac{u^2\sin^2\theta}{2g}}$

- **Horizontal Range –** It is the distance covered by the object between its point of projection and the point of hitting the ground and is denoted by *R*.

  R is the distance travelled during the time of flight *T*.

  $\therefore$ $R = u\cos\theta \times T = u\cos\theta \times 2\left(\frac{u\sin\theta}{g}\right)$

  $\Rightarrow$ $R = \frac{u^2}{g}(2\sin\theta\cos\theta)$

  $\Rightarrow$ $\boxed{R = \frac{u^2\sin 2\theta}{g}}$

  For the maximum horizontal range,

  sin 2θ = 1 = sin 90°

  $\Rightarrow$ 2θ = 90°

  $\Rightarrow$ θ = 45°

  $\therefore$ Maximum horizontal range,

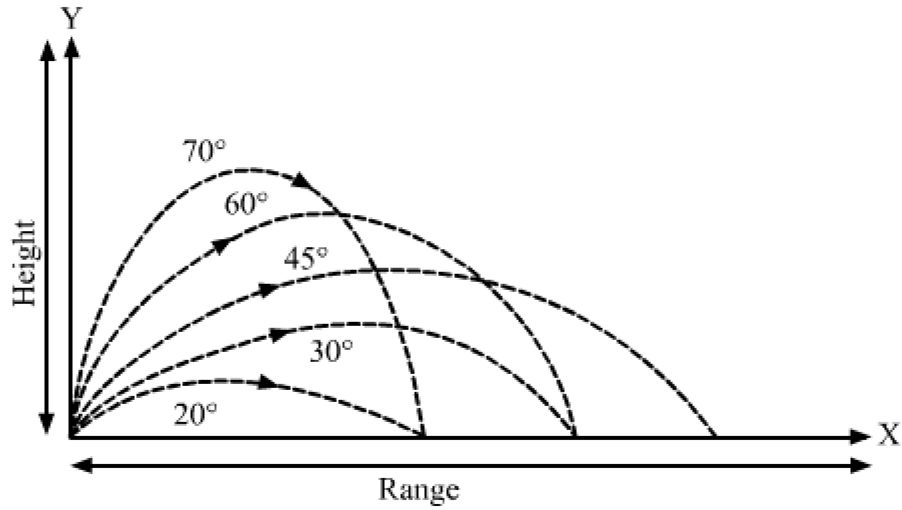
  $$R_m = \frac{u^2}{g}\sin 2\times 45° = \frac{u^2}{g}$$

- **Angle of projection -** It is the angle made by the velocity of projection with the horizontal.

  Angle of projection for projectile motion,

$$\theta_o = \tan^{-1}\left(\frac{4H}{R}\right)$$

The figure shows the trajectories of the projectile at different angles of projection.



Case I

When the angle of projection is 45°, the height of projectile,

$$H = \frac{u^2 \sin^2\theta}{2g}$$

$$\Rightarrow H = \frac{u^2 \sin^2 45^\circ}{2g}$$

$$\therefore H = \frac{u^2}{4g} \left(\because \sin 45^\circ = \frac{1}{\sqrt{2}}\right)$$

Range of the projectile at 45°,

$$R_{max} = \frac{u^2}{g}$$

$$\therefore R_{max} = 4H_{max}$$

Case II

When the angle of projection is 90°, height,

$$H = \frac{u^2 \sin^2\theta}{2g}$$

$$\Rightarrow H = \frac{u^2 (\sin 90^\circ)^2}{2g}$$

$$\Rightarrow H = \frac{u^2}{2g}$$

When angle of projection is 90°, range,

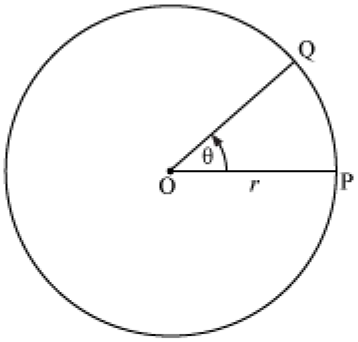
$$R = \frac{u^2 \sin 2\theta}{g}$$

$$\Rightarrow R = \frac{u^2 \sin 180^\circ}{g}$$

$$\Rightarrow R = 0$$

Uniform Circular Motion

- **Angular displacement** (θ) - It is the angle traced out by the radius vector at the centre of a circular path in a given time.



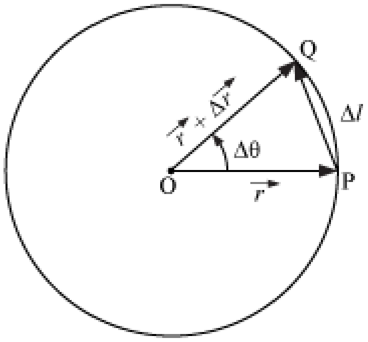
$$\theta = \frac{PQ}{r} \quad \left(\because \text{Angle} = \frac{\text{Arc}}{\text{Radius}}\right)$$

- **Angular velocity** (ω) - It is the rate of change of angular displacement.

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

ω=limΔt→0ΔθΔt=dθdt

- **Relation between linear velocity and angular velocity**



Let:
ω = Uniform angular velocity of the point object moving along PQ

v = Linear speed

r = Radius of the circular path

t = Time at which the object is at point P

($t + \Delta t$) = Time at which the object is at point Q

POQ = $\Delta\theta$

$\overrightarrow{OQ} = \vec{r} + \overrightarrow{\Delta r}$

It means that the object describes arc PQ of length Δl in time interval Δt.

$$\therefore v = \frac{\Delta l}{\Delta t}$$
$$\Delta l = v\Delta t \qquad \text{...(i)}$$

$$\omega = \frac{\Delta\theta}{\Delta t}$$
$$\Delta\,\theta = \omega\,\Delta\,t \qquad \ldots\text{(ii)}$$

Also,

$$\Delta\theta = \frac{\Delta l}{r} \qquad \left[\because \text{Angle} = \frac{\text{Arc}}{\text{Radius}}\right]$$

From equations (i) and (ii), we get:

$$\omega\Delta t = \frac{v\Delta t}{r}$$

$$\boxed{\therefore v = r\omega}$$

- **Direction of** $\vec{v}$ - Velocity at any point in circular motion is directed along the tangent to the circle at that point in the direction of motion.

- **Angular acceleration** (α) - It is the rate of change of angular velocity.

$$\alpha = \underset{\Delta t \to 0}{Lt} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

- **Relation between linear acceleration and angular acceleration**

  We know:
  $v = r\omega$

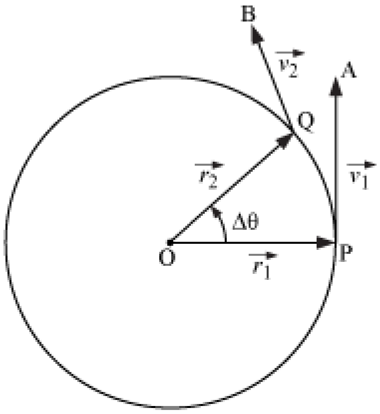
  Differentiating with respect to time, we get:

  $$\frac{dv}{dt} = \left(\frac{d\omega}{dt}\right) r$$

  $$\boxed{\therefore a = \alpha r}$$

- **Centripetal acceleration**

  - Acceleration acting on the object undergoing uniform circular motion is called centripetal acceleration.



  -
  - Consider a particle of mass *m* moving with a constant speed *v* and uniform angular velocity ω.
  - Let at any time, the particle be at point P, where $\overrightarrow{OP} = \vec{r_1}$, and at time $t + \Delta t$, the particle be at point Q, where $\overrightarrow{OQ} = \vec{r_2}$ and $\angle POQ = \Delta\theta$.
  - Also,
    $|\vec{r_1}| = |\vec{r_2}| = r$

  Now,

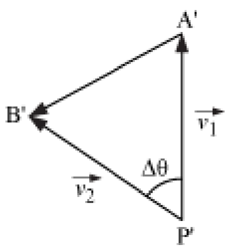
$$\omega = \frac{\Delta\theta}{\Delta t} \qquad \ldots \text{(i)}$$

Let $\vec{v_1}$ and $\vec{v_2}$ be the velocity vectors of the particle at locations P and Q, respectively.

We can represent $\vec{v_1}$ and $\vec{v_2}$ in magnitude and direction by tangents $\overrightarrow{PA}$ and $\overrightarrow{QB}$.

Because the particle is moving with a uniform speed v, the length of the tangents is equal, i.e, $|\overrightarrow{PA}| = |\overrightarrow{QB}| = |\vec{v}|$.

To find the change in velocity in time intervals t and $t + \Delta t$, take an external point P'. Now, draw $\overrightarrow{P'A'}$ and $\overrightarrow{P'B'}$ representing velocity vectors $\vec{v_1}$ and $\vec{v_2}$, respectively.



Clearly,

$\angle A'P'B' = \Delta\theta$

From the triangle law of vectors, we have:

$$\overrightarrow{P'A'} + \overrightarrow{A'B'} = \overrightarrow{P'B'}$$
$$\overrightarrow{A'B'} = \overrightarrow{P'B'} - \overrightarrow{P'A'} = \vec{v_2} - \vec{v_1} = \Delta\vec{v} \text{ (say)}$$

As $\Delta t \to 0$, A' lies close to B'. Now, $A'B'$ can be taken as an arc $A'B'$ of circle of radius $P'A' = |\vec{v}|$.

$$\therefore \Delta\theta = \frac{A'B'}{P'A'} = \frac{|\Delta\vec{v}|}{|\vec{v}|}$$

$$\omega\Delta t = \frac{|\Delta\vec{v}|}{|\vec{v}|}$$

$$\frac{|\Delta\vec{v}|}{\Delta t} = |\vec{v}|\omega = (r\omega)\omega = \omega^2 r \qquad [\because v = r\omega]$$

When $\Delta t \to 0$, $\frac{|\Delta \vec{v}|}{\Delta t}$ represents the magnitude of centripetal acceleration at P, which is given by

$$|\vec{a}| = \frac{|\Delta \vec{v}|}{\Delta t} = \omega^2 r = \left(\frac{v}{r}\right)^2 r = \frac{v^2}{r}$$

$$\therefore \boxed{|\vec{a}| = \omega^2 r = \frac{v^2}{r}}$$

Direction of centripetal acceleration - Centripetal acceleration vector acts along the radius of a circular path and is directed towards the centre of the circular path.

- **Centripetal Force**

It is a force that acts on a particle performing circular motion along the radius of a circle; it is directed towards the centre of the circle. It is a necessary force for maintaining circular motion.

Because centripetal force acts at right angles to the tangential velocity of a particle, there is no displacement in the direction of force. Hence, no work is done by centripetal force.
According to Newton's second law of motion,
Force = Mass × Acceleration
Here,
$F_{CP} = ma$

$$\because\ a = v\omega = \frac{v^2}{r} = r\omega^2$$

$$\therefore F_{CP} = mv\omega = \frac{mv^2}{r} = mr\omega^2$$

In vector notation,

$$\overrightarrow{F_{CP}} = -\frac{mv^2}{r}\hat{r} = -m\omega^2\vec{r}$$

Here,

m = Mass of the particle performing uniform circular motion

v = Linear speed of the particle performing uniform circular motion

r = Radius of the circle

$\vec{r}$ = Radius vector

ω = Angular speed of the particle performing uniform circular motion

$\hat{r}$ = Unit vector in the direction of $\vec{r}$

The SI unit of centripetal force is newton (N).

Examples of Centripetal force

1. When a car is travelling round a circular horizontal road with uniform speed, the necessary centripetal force for the circular turn is provided by the force of friction between the tyres of the car and the road.

2. When an object tied at the end of a string is whirled in a horizontal circle, the necessary centripetal force for maintaining circular motion is provided by the tension in the string.

3. For electrons revolving around the nucleus of an atom, the centripetal force is provided by the electrostatic force of attraction between the positively charged nucleus and the negatively charged electron.

4. The Moon revolves around the Earth due to necessary centripetal force provided by the gravitational force of attraction between the Moon and Earth.

- **Centrifugal Force**

Newton's laws of motion are valid only in the inertial frame of reference, i.e., non-accelerated frame of reference. In order to make Newton's laws of motion valid in the non-inertial frame of reference, we have to imagine a force called a pseudo force that is not real but appears so due to acceleration of frame of reference.
Such a pseudo force imagined in circular motion in order to make Newton's laws of motion valid is called centrifugal force.

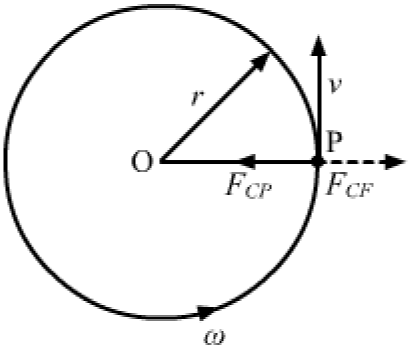
Centrifugal force is a pseudo force in uniform circular motion. It acts along the radius and is directed away from the centre of the circle.

Magnitude of centrifugal force = Mass × Acceleration of reference frame

$$F_{CF} = mv\omega = \frac{mv^2}{r} = mr\omega^2$$

In vector notation,

$$\overrightarrow{F_{CF}} = +\frac{mv^2}{r}\hat{r} = +m\omega^2\overrightarrow{r}$$



Here,

P = Particle performing uniform circular motion

$\overrightarrow{F_{CP}}$ = Centripetal force

$\overrightarrow{F_{CF}}$ = Centrifugal force

r = Radius of the circle

v = Linear speed of P

ω = Angular speed of P

Examples of Centrifugal Force

1. When a car in motion takes a sudden turn towards left, passengers in the car experience an outward push towards the right. This is due to centrifugal force acting on the passengers.

2. We experience an outward pull in merry-go-round when it rotates about the vertical axis. This is due to centrifugal force acting on us.

3. The bulging of the Earth at the equator and the flattening of the Earth at the poles are due to centrifugal force acting upon it.

4. Drier in a washing machine consists of perforated walls. As the cylindrical vessel is rotated fast, centrifugal force forces out water through perforations, thereby drying clothes quickly.

5. Centrifuge is a device used for separating heavy and light particles; it works on the principle of centrifugal force.